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Symmetric association schemes attached to finite upper half planes over rings

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Abstract

The finite upper half planes over finite fields and rings are finite analogues of the Poincaré upper half plane. The general linear group G acts transitively on the finite upper half plane. Let K denote the stabilizer of a point. In the case of fields, it is well-known that the pair of (G, K) is a Gelfand pair. In this paper, we show that (G, K) is also a Gelfand pair in the case of rings.

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1. Introduction

The finite upper half planes over finite fields and rings were investigated by many reseachers—Angel, Evans, Stark, Terras, and Trimble—to name a few. They are defined as finite analogues of the Poincaré upper half plane:

$$H = \{x + \mathrm{i}y \mid x, y \in \mathbb{R}, y > 0\}.$$

Similar to the Poincaré version, the general linear groups over finite fields or rings act transitively on the finite upper half planes by the fractional linear transformation. From these actions, we obtain association schemes. In the case of finite

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fields, all the relations of these association schemes are given by an analogue of the Poincaré distance. We see from this fact that the association scheme is symmetric, in particular, commutative. The commutativity of association schemes is a very important property since we can consider the first eigenmatrix, or equivalently, the zonal spherical functions only for commutative association schemes and since we can see graph theoretical information from the first eigenmatrix. For example we can calculate the structure constants and expansion constants from the first eigenmatrix. The first eigenmatrices of the finite upper half planes over finite fields have been calculated in [6,8].

A k-regular (undirected) graph is called a Ramanujan graph if any eigenvalue θ of the graph with $|\theta| \neq k$ satisfies $|\theta| \leqslant 2\sqrt{k-1}$. Since an association scheme is a family of some regular graphs with the same vertex set, we can obtain many regular graphs from association schemes, in fact, we obtain many examples of Ramanujan graphs from association schemes (see [4,5] for such examples). The finite upper half planes over finite fields are such examples except only for the two trivial relations of the association scheme, that is, they become Ramanujan graphs (see [10]). In [2], the finite upper half planes over $\mathbb{Z}/p^r\mathbb{Z}$ are introduced and studied for an odd prime p. In that paper, it is proved that the finite upper half planes are not Ramanujan graphs for $r = 2$ and $p \geqslant 5$. They studied the structure of the graphs extensively. However, it seems that it has not been known yet whether the association scheme obtained from the action of the general linear group is symmetric and whether the graphs obtained from the association scheme are Ramanujan graphs for $r \geqslant 3$. In this paper, we shall show that the association scheme is symmetric (that is, commutative), and then, we shall discuss how the relations are given. The purpose of this paper is to start a research for the finite upper half planes over rings from the viewpoint of association schemes.

2. Association scheme

In this section, we shall introduce an association scheme. The concept of association schemes plays a very important role in algebraic combinatorics since it gives us a unified viewpoint and way for various combinatorial objects, e.g. coding theory, design theory, algebraic graph theory, group theory. We want to consider the upper half plane from the viewpoint of association schemes. The reader is referred to [3] for the basic theory of association schemes.

Definition 2.1 (*Association scheme*). Let X be a finite set and $X \times X = \dot{\bigcup}_{i=0\ldots d} R_i$ where $\dot{\bigcup}$ means a disjoint union. We call a pair $(X, \{R_i\}_{i=0\ldots d})$ an association scheme of class d if X and $\{R_i\}_{i=0\ldots d}$ satisfy the following properties (i)–(iii):

(i) $R_0 = \{(x, x) \mid x \in X\}$,
(ii) ${}^tR_i = R_{i'}$ for some i' where ${}^tR_i := \{(y, x) \mid (x, y) \in R_i\}$,

(iii) For $(x, y) \in R_k$, the number of z with $(x, z) \in R_i$ and $(z, y) \in R_j$ does not depend on a choice of $(x, y) \in R_k$. Denote this constant number by p_{ij}^k.
Moreover, if the following property (iv) or (v) holds, we say that the association scheme is commutative or symmetric respectively:
(iv) $p_{ij}^k = p_{ji}^k$,
(v) $R_i = {}^t R_i$.

It is easy to see that a symmetric association scheme is a commutative association scheme.

Let X be a finite set and G a group acting on X from the left side transitively. Then G acts naturally on $X \times X$ also, i.e. for $(x, y) \in X \times X$ and $g \in G$, the action of G on $X \times X$ is defined to be $g \cdot (x, y) := (g \cdot x, g \cdot y)$ where $g \cdot x$ denotes the action of G on X. The orbits of $X \times X$ are called 2-orbits of the action of G on X. It is easy to see that the pair of X and the 2-orbits constructs an association scheme. Let H be the stabilizer of a point in X. G acts transitively on the right cosets G/H by the multiplication from the left side. Then G/H and X are isomorphic as G-sets and so are $X \times X$ and $G/H \times G/H$. From the above we can reduce the problem of 2-orbits to that of a pair of a group G and a subgroup H.

In general, for a group G and a subgroup H of G, we denote the association scheme obtained from the action of G on the right cosets G/H by $\mathfrak{X}(G, H)$. Many association schemes are obtained in this way but not all. Association schemes which are not obtained from a group action are called non-Schurian. See [7,11] for such examples.

Definition 2.2 (*Gelfand pair*). Let G be a group and H a subgroup of G. A pair (G, H) is called a Gelfand pair when the association scheme $\mathfrak{X}(G, H)$ is commutative.

Though Definition 2.2 is different from that given in [10], we can check the equivalence of these definitions. The main result in this paper is that a pair of the general linear group and the stabilizer of a point by the action on the finite upper half plane is a Gelfand pair.

3. Definitions and lemmas

First we prepare the notation used throughout this paper. We denote the group of units of a ring R by $U(R)$. Let p be an odd prime. In brief, we write (p) instead of mod p.

Let $R_n := \mathbb{Z}/p^n\mathbb{Z}$ for a natural number n. Let U_n be the group of units, i.e. $U_n := U(R_n) = \{x \in R_n \mid x \not\equiv 0 \ (p)\}$. Clearly $|U_n| = p^{n-1}(p-1)$. Since U_n is a cyclic group, we can take a generator δ of U_n. For convenience, we put $R_0 = U_0 := \{0\}$.

Next we define an extension ring of R_n. Let M_n be the free R_n-module of rank 2 with the formal basis $\{1, \sqrt{\delta}\}$. That is,

$$M_n := \{x + y\sqrt{\delta} \mid x, y \in R_n\}.$$

Define the multiplication in M_n naturally as follows:
For $x_1 + y_1\sqrt{\delta},\ x_2 + y_2\sqrt{\delta} \in M_n$,

$$(x_1 + y_1\sqrt{\delta}) \cdot (x_2 + y_2\sqrt{\delta}) := x_1x_2 + y_1y_2\delta + (x_1y_2 + x_2y_1)\sqrt{\delta}.$$

Then, M_n becomes a commutative ring containing R_n. We regard M_n as an analogue of the complex plane.

In a similar way to the complex plane, we define the real part, the imaginary part and the conjugation:

$$\mathrm{Re}(z) := x, \quad \mathrm{Im}(z) := y, \quad \overline{z} := x - y\sqrt{\delta} \quad \text{for } z = x + y\sqrt{\delta} \in M_n, \text{ respectively.}$$

We also define a norm $N_n : M_n \longrightarrow R_n$ as follows:

$$N_n(z) := x^2 - y^2\delta \quad \text{for } z = x + y\sqrt{\delta} \in M_n.$$

N_n preserves the multiplication of M_n: $N_n(zw) = N_n(z)N_n(w)$ for $z, w \in M_n$.

Lemma 3.1. $U(M_n) = \{z \in M_n \mid N_n(z) \in U_n\}$.

Proof. It follows directly from the fact that $z^{-1} = \overline{z}/N_n(z)$. □

Lemma 3.2. *N_n maps $U(M_n)$ onto U_n.*

Proof. Let $Q_n := \{x^2 \mid x \in U_n\}$ and $NQ_n := U_n - Q_n$. Clearly we have $Q_n \subset N_n(U(M_n))$. Assume that there exists some $z = x + y\sqrt{\delta} \in U(M_n)$ such that $N_n(z) \in NQ_n$. Then, since $N_n(rz) = (rx)^2 - (ry)^2\delta = r^2(x^2 - y^2\delta) = r^2N_n(z)$, we get $N_n(U(M_n)) = U_n$. So when r runs through U_n, $N_n(rz)$ moves through NQ_n. Therefore we only have to verify the existence of such z. Also we know that $r \in Q_n$ if and only if $r \pmod{p} \in Q_1$ for $r \in U_n$. So we can reduce this existence to the case of U_1. But this existence is verified simply. □

The finite upper half plane over field was defined as a subset of the extension field $\mathbb{F}_{q^2}$ of the finite field $\mathbb{F}_q$ where $\mathbb{F}_q$ denotes the field with q elements (see [1,10]) . In a similar way, we shall define the finite upper half plane over ring as a subset of M_n defined above (see [2]).

Definition 3.1 (*The finite upper half planes over finite rings*).

$$\mathbb{H} := \{x + y\sqrt{\delta} \in M_n \mid x \in R_n, y \in U\}.$$

Next we shall consider the general linear group over R_n. Let

$$G_n := \mathrm{GL}(2, R_n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \, a, b, c, d \in R_n,\ ad - bc \not\equiv 0\ (p) \right\}.$$

Clearly $|G_n| = p^{4n-3}(p-1)^2(p+1)$ and the center $Z(G_n)$ of G_n consists of all scalar matrices such that reducing modulo p are not zero. That is to say,

$$Z(G_n) = \{aI \mid a \in U_n\}.$$

The following lemma follows [2].

Lemma 3.3. *Let $z \in \mathbb{H}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_n$. Then,*

(i) $cz + d \in U(M_n)$,
(ii) $(az + b)/(cz + d) \in \mathbb{H}$.

From Lemma 3.3 we see that G_n acts on $\mathbb{H}$ by the fractional linear transformation: $g \cdot z := \frac{az+b}{cz+d}$ for $z \in \mathbb{H}$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_n$. To see that this action is transitive, we consider the affine group A_n:

$$A_n := \left\{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mid y \in U_n,\ x \in R_n \right\}.$$

Clearly $|A_n| = |\mathbb{H}| = p^{2n-1}(p-1)$. Since $\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \cdot \sqrt{\delta} = x + y\sqrt{\delta}$, A_n acts transitively on $\mathbb{H}$ and so is G_n.

Next we consider the stabilizer of $\sqrt{\delta}$ in G_n. This subgroup is called the orthogonal group by a comparison to the case of the Poincaré upper half plane.

$$K_n := \{g \in G \mid g \cdot \sqrt{\delta} = \sqrt{\delta}\} = \left\{ \begin{pmatrix} a & b\delta \\ b & a \end{pmatrix} \mid a, b \in R_n,\ a^2 - b^2\delta \not\equiv 0\ (p) \right\}.$$

Clearly $|K_n| = p^{2(n-1)}(p+1)(p-1)$. K_n is isomorphic to $U(M_n)$ by the correspondence: $\begin{pmatrix} a & b\delta \\ b & a \end{pmatrix} \longmapsto a + b\sqrt{\delta}$. In the case of fields, the orthogonal group is cyclic, but it is not true in the case of rings. The following proposition shows that the case of rings is more difficult than the case of fields.

Proposition 3.1. $U(M_n) \simeq \mathbb{Z}/p^{n-1}\mathbb{Z} \oplus \mathbb{Z}/p^{n-1}(p^2-1)\mathbb{Z}$.

We omit the proof of this proposition because we do not need it in this paper. Below we omit the index n, e.g. we write R instead of R_n.

We shall determine the double coset decomposition $K\backslash G/K$. Clearly G is decomposed into the right cosets by K as follows:

$$G = \sum_{x \in R,\ y \in U} \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} K = \sum_{z \in A} zK.$$

So we can take all elements of A as the complete set of representatives of $K\backslash G/K$. From now on, we identify $\mathbb{H}$ with G/K or A as G-sets by the following correspondences:

$$x + y\sqrt{\delta} \longleftrightarrow \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} K \longleftrightarrow \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}.$$

Also by the above correspondences we regard functions on $\mathbb{H}$ as those on G/K or A. For example, we write

$$\mathrm{Im}(x + y\sqrt{\delta}) = \mathrm{Im}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} K\right) = \mathrm{Im}\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}$$

and so on.

In order to obtain the criterion whether two elements of A are in the same class of the double coset decomposition, we introduce finite analogues of the Poincaré distance. This distance follows [2].

Definition 3.2. For $z, w \in \mathbb{H}$, we define the distance d between z and w by

$$d(z, w) := \frac{N(z - w)}{\mathrm{Im}(z)\,\mathrm{Im}(w)}.$$

This distance d is invariant under the action of G. That is to say, $d(g \cdot z, h \cdot w) = d(z, w)$ for any $g, h \in G$ and any $z,\ w \in \mathbb{H}$. For each $a \in R_n$, we put

$$S(a) := \{g \in A \mid d(\sqrt{\delta}, g \cdot \sqrt{\delta}) = a\}.$$

Since d is invariant under the action of G, $S(a)$ is a sum of some double cosets. Therefore we only have to investigate the decomposition of each $S(a)$.

4. The criterion of the double cosets and the symmetry of the association scheme

In this section, we shall consider the criterion whether two elements of A are in the same class, and we shall show that the association scheme is symmetric as a corollary. The following lemma is simple, but very useful in this paper.

Lemma 4.1. *Let $c \in U_n$. Then,*

$$\sharp\{(x, y) \mid x, y \in R_n,\ x^2 - y^2\delta = c\} = p^{n-1}(p + 1).$$

Proof. Note that $x^2 - y^2\delta$ is the norm of $x + y\sqrt{\delta}$. From Lemma 3.2, the number of solutions of $x^2 - y^2\delta = c$ for any element c in U_n is equal to that of $x^2 - y^2\delta = 1$. So it is equal to $|U(M_n)|/|U_n|$. $\square$

In the case of fields, $S(a)$ is a double coset in itself (note that we are identifying A with G/K). But in the case of rings, $S(a)$ is not a double coset in general. But the following theorem holds for some distances and follows [2].

Theorem 4.1. *Let $a \in R_n$ such that $a \not\equiv 0, 4\delta\ (p)$. Then $S(a)$ is a double coset.*

Proof. For $g \in K$ and $z \in S(a)$, $g \cdot z = z$ if and only if g is an element of $Z(G_n)$ (note $a \not\equiv 0, 4\delta\ (p)$). So $Z(G_n)$ is the stabilizer of z. We denote the K-orbit of z by Kz. Then, $|Kz| = |K_n|/|Z(G_n)| = p^{n-1}(p+1)$. While, we see by elementary calculation that for $z = x + y\sqrt{\delta} \in \mathbb{H}$,

$$z \in S(a) \Longleftrightarrow x^2 \equiv \delta \left(y + \frac{a-2\delta}{2\delta} \right)^2 - \frac{a(a-4\delta)}{4\delta}\ (p^n). \tag{1}$$

From Lemma 4.1, the number of solutions (x, y) to (1) is equal to $p^{n-1}(p+1)$. Therefore $|S(a)| = |Kz| = p^{n-1}(p+1)$. This shows that $S(a) = Kz$. $\square$

For $a \equiv 0, 4\delta\ (p) \in R_n$, $S(a)$ is decomposed into some double cosets. To see this fact, we define the p-adic valuation.

Definition 4.1 (*p-adic valuation*). Let $a \in R_n$. If $a \neq 0$ and $p^l \parallel a$ (i.e. $p^l \mid a$ and $p^{l+1} \nmid a$), we define $h(a) := l$. If $a = 0$, we define $h(a) := n$. We call h the p-adic valuation of R_n.

We obtain the following lemma by using the p-adic valuation.

Lemma 4.2. *Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(R)$ and h the p-adic valuation of R_n. Assume that A is a nonzero matrix. We put $h := \min_{1 \leqslant i,j \leqslant 2} h(a_{ij})$, $a_{ij} = p^h a'_{ij}$ and $A' := \begin{pmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{pmatrix}$ where $a'_{ij} \in R_{n-h}$. Then, there exits a vector $(x, y) \not\equiv (0, 0)\ (p)$ such that $A \begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}\ (p^n)$ if and only if* $\det A' \equiv 0\ (p^{n-h})$.

We leave the proof of Lemma 4.2 to the reader.

Let us consider the criterion whether two elements of $\mathbb{H}$ are in a same double coset. First we consider $S(a)$ such that $a \equiv 0\ (p)$.

Let $a \equiv 0\ (p)$ and $g = \begin{pmatrix} 1+l & k \\ 0 & 1 \end{pmatrix} \in S(a)$. Then, we write $h(g) = \min\{h(l), h(k)\}$ using Definition 4.1. Since $g \in S(a)$, $d\left(\sqrt{\delta}, \begin{pmatrix} 1+l & k \\ 0 & 1 \end{pmatrix} \cdot \sqrt{\delta}\right) = \frac{k^2 - l^2\delta}{1+l} = a$. Since $a \equiv 0\ (p)$ and δ is a nonsquare, we see that k and l are in pR_n.

Take $g_1 = \begin{pmatrix} 1+l_1 & k_1 \\ 0 & 1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} 1+l_2 & k_2 \\ 0 & 1 \end{pmatrix}$ in $S(a)$ where k_i and l_i are in pR_n. Then, by elementary calculation, we see that $Kg_1K = Kg_2K$ if and only if there exists a vector $(a, b) \not\equiv (0, 0)$ (p) such that

$$\begin{pmatrix} k_2 - k_1 & \delta(l_1 + l_2) + (k_1k_2 + l_1l_2\delta) \\ l_2 - l_1 & (k_1 + k_2) + (l_1k_2 + k_1l_2) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \ (p^n). \tag{2}$$

It is easy to see that the minimum value of the p-adic valuations of entries of the coefficient matrix in (2) is equal to $h := \min\{h(g_1),\ h(g_2)\}$. So when we put $l_i = l_i' p^h$, and $k_i = k_i' p^h$ where $l_i, k_i \in R_{n-h}$, from Lemma 4.2, Eq. (2) is equivalent to

$$\frac{N_{n-h}(k_1' + l_1'\sqrt{\delta})}{1 + l_1} \equiv \frac{N_{n-h}(k_2' + l_2'\sqrt{\delta})}{1 + l_2} \ (p^{n-h}). \tag{3}$$

Similarly for $a \equiv 4\delta$ (p), we take $g := \begin{pmatrix} -1+l & k \\ 0 & 1 \end{pmatrix} \in S(a)$. Then, we write $h(g) = \min\{h(l),\ h(k)\}$. Since $g \in S(a)$, $d\left(\sqrt{\delta}, \begin{pmatrix} -1+l & k \\ 0 & 1 \end{pmatrix} \cdot \sqrt{\delta}\right) = \frac{k^2 - l^2\delta}{-1+l} = a$. Since $a \equiv 4\delta$ (p) and δ is a nonsquare, we see that k and l are in pR_n.

Take $g_1 = \begin{pmatrix} -1+l_1 & k_1 \\ 0 & 1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} -1+l_2 & k_2 \\ 0 & 1 \end{pmatrix} \in S(a)$ where k_i and $l_i \in pR_n$. Then, similar to when $a \equiv 0$ (p), we see that $Kg_1K = Kg_2K$ if and only if

$$\frac{N_{n-h}(k_1' + l_1'\sqrt{\delta})}{-1 + l_1} \equiv \frac{N_{n-h}(k_2' + l_2'\sqrt{\delta})}{-1 + l_2} \ (p^{n-h}). \tag{4}$$

Using (3) and (4), we define the two new functions $H : \cup_{a\equiv 0,4\delta\ (p)} S(a) \longrightarrow \{0, 1, \ldots, n\}$ and $L_H : \cup_{a\equiv 0,4\delta\ (p)} S(a) \longrightarrow \bigcup_{i=0}^{n} U_i$ for $a \equiv 0,\ 4\delta$ (p). We divide into the cases $a \equiv 0$ (p) and $a \equiv 4\delta$ (p).

(i) $a \equiv 0$ (p)

Let $g = \begin{pmatrix} 1+l & k \\ 0 & 1 \end{pmatrix} \in S(a)$. Put $H(g) := \min\{h(l),\ h(k)\}$ and $k = k' p^{H(g)}$, $l = l' p^{H(g)}$ where $k', l' \in U_{n-H(g)}$. Then, we define

$$L_H(g) := \frac{N_{n-h}(k' + l'\sqrt{\delta})}{1 + l}.$$

Note that $L_H(g)$ is in $U_{n-H(g)}$.

(ii) $a \equiv 4\delta$ (p)

Let $g = \begin{pmatrix} -1+l & k \\ 0 & 1 \end{pmatrix} \in S(a)$. Put $H(g) := \min\{h(l),\ h(k)\}$ and $k = k' p^{H(g)}$, $l = l' p^{H(g)}$ where $k', l' \in U_{n-H(g)}$. Then, we define

$$L_H(g) := \frac{N_{n-h}(k' + l'\sqrt{\delta})}{-1 + l}.$$

Note that $L_H(g) \in U_{n-H(g)}$.

From the above, we obtain the following theorem.

Theorem 4.2. *Let $a \equiv 0$ or $4\delta\ (p)$, and $g_1, g_2 \in S(a)$. Then, $Kg_1K = Kg_2K$ if and only if $H(g_1) = H(g_2)$ and $L_H(g_1) = L_H(g_2)$.*

Since we can verify easily that the above criterion is symmetric, we obtain the following corollary directly.

Corollary 4.1. *$\mathfrak{X}(G, K)$ is a symmetric association scheme.*

Also by using Theorem 3.2 we can decompose G into the double cosets with respect to K as follows:

$$G = \sum_{a \not\equiv 0,\ 4\delta\ (p)} Kg_aK + \sum_{\substack{1 \leqslant h \leqslant n-1 \\ u \in U_{n-h} \\ i=0 \text{ or } 4\delta}} Kg^i_{h,u}K + K\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}K + K.$$

In the first sum, g_a is an element in $S(a)$ for each $a \not\equiv 0, 4\delta\ (p)$. In the second sum, for $1 \leqslant h \leqslant n-1$, $u \in U_{n-h}$ and $i = 0, 4\delta$, $g^i_{h,u}$ is an element of A such that $d(\sqrt{\delta}, g^i_{h,u}\sqrt{\delta}) = i\ (p)$, $H(g^0_{h,u}) = H(g^{4\delta}_{h,u}) = h$, and $L_H(g^0_{h,u}) = L_H(g^{4\delta}_{h,u}) = u$. Lemma 4.1 secures the existence of such $g^i_{h,u}$.

Therefore the number of double cosets (the rank of $\mathfrak{X}(G, K)$) is

$$p^n - 2p^{n-1} + 2p^{n-1} = p^n.$$

Note that this number is equal to the cardinality of R_n.

5. The modified distance

In this section, we shall modify the distance d so that the modified distance d_R gives the relations of $\mathfrak{X}(G, K)$. For $z, w \in \mathbb{H}$, we define the modified distance d_R as follows:

(i) If $d(z, w) \not\equiv 0, 4\delta\ (p)$, then we define $d_R(z, w) := d(z, w)$.

(ii) If $d(z, w) \equiv 0\ (p)$. Put $r := \mathrm{Re}(z - w)$, $k := \mathrm{Im}(z - w)$, and $h := \min\{h(r), h(k)\}$, moreover write $r = p^h r'$, $k = p^h k'$, where $r', k' \in R_{n-h}$. Then, we define

$$d_R(z, w) := \frac{N_{n-h}(r' + k'\sqrt{\delta})p^h}{\mathrm{Im}(z)\,\mathrm{Im}(w)}.$$

$N_{n-h}(r' + k'\sqrt{\delta})$ is in R_{n-h}, but since we multiply p^h by $N_{n-h}(r' + k'\sqrt{\delta})$, we can regard $d_R(z, w)$ as an element of R_n.

(iii) If $d(z, w) \equiv 4\delta\ (p)$. Put $r := \mathrm{Re}(z - w)$, $k := \mathrm{Im}(z + w)$, and $h := \min\{h(r), h(k)\}$, moreover write $r = p^h r'$, $k = p^h k'$, where $r', k' \in R_{n-h}$. Then, we define

$$d_R(z, w) := \frac{N_{n-h}(r' + k'\sqrt{\delta})p^h}{\mathrm{Im(z)\,Im(w)}} + 4\delta$$

In a similar way to (ii), we can regard $d_R(z, w)$ as an element of R_n.

For $a \in R_n$, we define $L_a := \{(z, w) \mid z, w \in \mathbb{H}, d_R(z, w) = a\}$. Then, $\{L_a\}_{a \in R_n}$ give the relations of $\mathfrak{X}(G, K)$. This fact follows directly from Theorem 4.2. Using Lemma 4.1, we can calculate the valency of L_a.

Fix $z \in \mathbb{H}$, and put $v_a := \sharp\{w \in \mathbb{H} \mid (z, w) \in L_a\}$. Then,

$$v_a = \begin{cases} p^{n-1}(p+1) & \text{if } a \not\equiv 0,\ 4\delta\ (p), \\ p^{n-h-1}(p+1) & \text{if } a = p^h u \text{ or } p^h u + 4\delta, \\ 1 & \text{if } a = 0 \text{ or } 4\delta, \end{cases}$$

where $1 \leqslant h \leqslant n - 1,\ u \in U_{n-h}$.

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References

[1] J. Angel, N. Celniker, S. Poulos, A. Terras, C. Trimble, E. Velasquez, Special functions on finite upper half planes, Contemp. Math. 138 (1992) 1–26.

[2] J. Angel, B. Shook, A. Terras, C. Trimble, Graph spectra for finite upper half planes over rings, Linear Algebra Appl. 226–228 (1995) 423–457.

[3] E. Bannai, T. Ito, Algebraic Combinatorics I, Association Schemes, Benjamin/Cummings, Menlo Park, CA, 1984.

[4] E. Bannai, O. Shimabukuro, H. Tanaka, Finite Euclidean graphs and Ramanujan graphs, Discrete Math., to appear.

[5] E. Bannai, O. Shimabukuro, H. Tanaka, Finite analogues of non-Euclidean spaces and Ramanujan graphs, European Journal of Combinatorics, in press.

[6] R. Evans, Characters sums as orthogonal eigenfunctions of adjacency operaters for Cayley graphs, Contemp. Math. 168 (1994) 33–50.

[7] S.S. Shrikhande, The uniqueness of the L_2 association schemes, Ann. Math. Stat. 30 (1959) 781–788.

[8] J. Soto-Andrade, Geometrical Gelfand models, tensor quotients, and Weil representations, in: Proc. Symp. Pure Math., 47, Amer. Math. Soc., Providence, 1987, pp. 305–316.

[9] M. Suzuki, Finite Simple Groups, Kinokuniya, 1987.

[10] A. Terras, Fourier Analysis on Finite Groups and Applications, London Math. Soc., 1999.

[11] H. Wielandt, Finite Permutation Groups, Academic Press, NY, 1964.